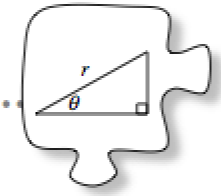
5.1.1 What if I know the hypotenuse?

Sine and Cosine Ratios

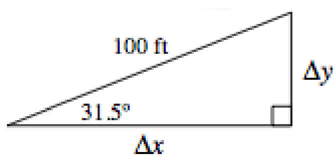


In the previous chapter, you used the idea of similarity in right triangles to find a relationship between the acute angles and the lengths of the legs of a right triangle. However, we do not always work just with the legs of a right triangle–sometimes we only know the length of the hypotenuse. By the end of today's lesson, you will be able to use two new trigonometric ratios that involve the hypotenuse of right triangles.

5-1. THE STREETS OF SAN FRANCISCO

While traveling around the beautiful city of San Francisco, Juanisha climbed several steep streets. One of the steepest, Filbert Street, has a slope angle of 31.5° according to her guidebook.

Once Juanisha finished walking 100 feet up the hill, she decided to figure out how high she had climbed. Juanisha drew the diagram below to represent this situation.

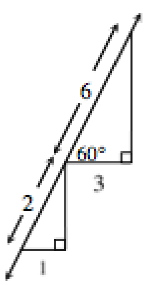


Juanisha's Drawing

Can a tangent ratio be used to find Δy? Why or why not? Be prepared to share your thinking with the rest of the class.

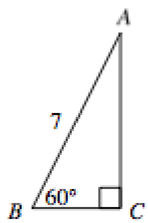
5-2. In order to find out how high Juanisha climbed in problem 5-1, you need to know more about the relationship between the ratios of the sides of a right triangle and the slope angle.

a. Use two different strategies to find Δy for the slope triangles shown in the diagram at below.



b. Find the ratio $\frac{\Delta x}{\text{hypotenuse}}$ for each triangle. Why must these ratios be equal?

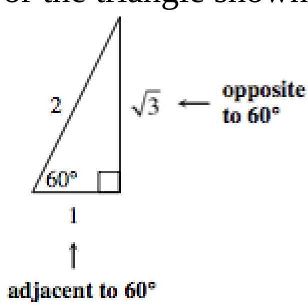
c. Find *BC* and *AC* in the triangle below. Show all work.



5-3. NEW TRIG RATIOS

In problem 5-2, you used a ratio that included the hypotenuse of ΔABC. There are several ratios that you might have used. One of those ratios is known as the **sine ratio** (pronounced "sign"). This is the ratio of the length of the side opposite the acute angle to the length of the hypotenuse.

For the triangle shown at right, the sine of 60° is $\frac{\sqrt{3}}{2} \approx 0.866$. This is written:



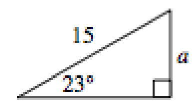
$$\sin 60° = \frac{\sqrt{3}}{2}$$

Another ratio comparing the length of the side adjacent to (which means "next to") the angle to the length of the hypotenuse, is called the **cosine ratio** (pronounced "co-sign"). For the triangle above, the cosine of 60° is $\frac{1}{2} = 0.5$. This is written:

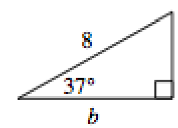
$$\cos 60° = \frac{1}{2}$$

a. Like the tangent ratio, your calculator can give you both the sine and cosine ratios for any angle. Locate the "sin" and "cos" buttons on your calculator and use them to find the sine and cosine of 60°. Does your calculator give you the correct ratios?

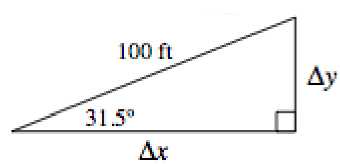
b. Use a trig ratio to write an equation and solve for *a* in the diagram at right. Does this require the sine ratio or the cosine ratio?



c. Likewise, write an equation and solve for *b* for the triangle at right.

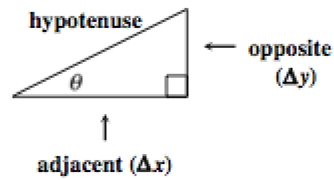


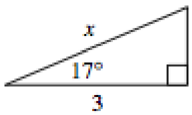
5-4. Return to the diagram from Juanisha's climb in problem 5-1. Juanisha still wants to know how many feet she climbed vertically when she walked up Filbert Street. Use one of your new trig ratios to find how high she climbed.

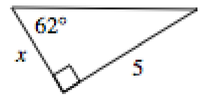


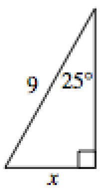
Juanisha's Drawing

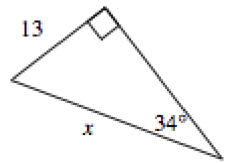
5-5. For each triangle below, decide which side is *opposite* and which is *adjacent* to the given acute angle. Then determine which of the three trig ratios will help you find x. Finally, write and solve an equation.

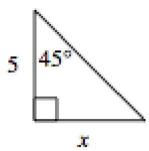


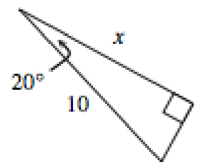
a. 

d. 

b. 

e. 

c. 

f. 

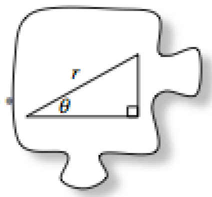
5-6. TRIANGLE TOOLKIT

Obtain a Lesson 5.1.1 Resource Page (“Triangle Toolkit”) from your teacher. This will be a continuation of the Geometry Toolkit you started in Chapter 1. Think about the tools you have developed so far to solve for the measure of sides and angles of a triangle. Then, in the space provided, add a diagram and a description of each tool you know. In later lessons, you will continue to add new triangle tools to this toolkit, so be sure to keep this resource page in a safe place. At this point, your toolkit should include:

- Pythagorean Theorem
- Tangent
- Sine
- Cosine

5.1.2 Which tool should I use?

Selecting a Trig Tool



You now have several tools that will help you find the length of a side of a right triangle when given any acute angle and any side length. But how do you know which tool to use? And how can you identify the relationships between the sides and the given angle?

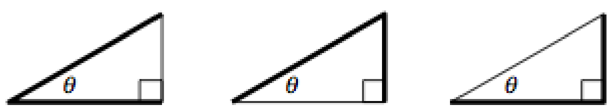
Today you will work with your team to develop **strategies** that will help you identify if cosine, sine, or tangent can be used to solve for a side of a right triangle. As you work, be sure to share any shortcuts you find that can help others identify which tool to use. As you work, keep the focus questions below in mind.

Is this triangle familiar? Is there something special about this triangle?

Which side is opposite the given angle? Which is adjacent?

Which tool should I use?

5-13. Obtain the Lesson 5.1.2 Resource Page from your teacher. On it, find the triangles shown below. Note: the diagrams are not drawn to scale. With your study team:



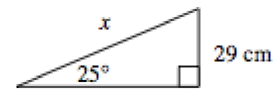
$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\tan\theta = \frac{\text{opp}}{\text{adj}}$

Look through all the triangles first and see if any look familiar or are ones that you know how to answer right away without using a trigonometric tool.

Then, for all the other triangles, identify which tool you should use based on where the **reference angle** (the given acute angle) is located and which side lengths are involved.

Write and solve an equation to find the missing side length.

5-14. Marta arrived for her geometry test only to find that she forgot her calculator. She decided to complete as much of each problem as possible.



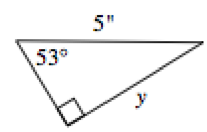
a. In the first problem on the test, Marta was asked to find the length x in the triangle shown at right. Using her algebra skills, she wrote and solved an equation. Her work is shown below. Explain what she did in each step.

$$\begin{aligned} \sin 25^\circ &= \frac{29}{x} \\ x(\sin 25^\circ) &= 29 \\ x &= \frac{29}{\sin 25^\circ} \end{aligned}$$

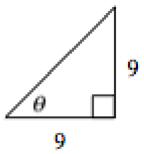
b. Marta's answer in part (a) is called an exact answer. Now use your calculator to help Marta find the approximate length of x.

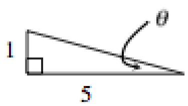
c. Marta's teammate, Ziv, said he solved it differently but still got the same answer. He started with the equation $\cos(65^\circ) = \frac{29}{x}$. Explain why this equation must give the same answer.

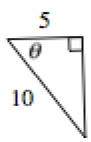
d. Solve for y in the diagram at right two ways, using both sine and cosine ratios. Make sure both strategies result in the same answer.



5-15. In problem 5-13, you used trigonometric tools to find a side length. But do you have a way to find an angle? Examine the triangles below. Do any of them look familiar? How can you use information about the side lengths to help you figure out the reference angle (*q*)? Your Trig Table Toolkit from Chapter 4 may be useful.

a. 

b. 

c. 

5-16. LEARNING LOG

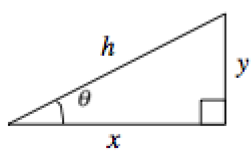
Write a Learning Log entry explaining how you know which trigonometric tool to use. Be sure to include examples with diagrams and anything else that would be useful to refer to later. Title this entry, "Choosing a Trig Tool" and label it with today's date.

METHODS AND MEANINGS

MATH NOTES

Trigonometric Ratios

You now have three **trigonometric ratios** you can use to solve for the missing side lengths and angle measurements in any right triangle. In the triangle below, when the sides are described relative to th angle θ, the opposite leg is y and the adjacent leg is x. The hypotenuse is h regardless of which acute angle is used.

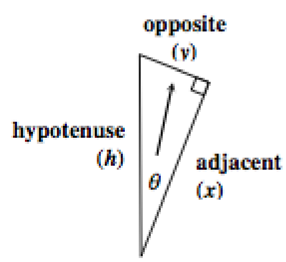


$$\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{y}{x}$$

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{y}{h}$$

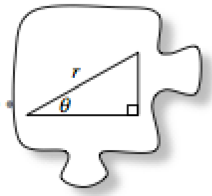
$$\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{x}{h}$$

In some cases, you may want to rotate the triangle so that it looks like a slope triangle in order to easily identify the reference angle θ, the opposite leg y, the adjacent leg x, and the hypotenuse h. Instead of rotating the triangle, some people identify the opposite leg as the leg that is always opposite (not touching) the angle. For example, in the diagram below, y is the leg opposite angle θ.



5.1.3 How can I find the angle?

Inverse Trigonometry



You now know how to find the missing side lengths in a right triangle given an acute angle and the length of any side. But what if you want to find the measure of an angle? If you are given the lengths of two sides of a right triangle, can you work backwards to find the measurements of the unknown angles? Today you will work on "undoing" the different trigonometric ratios to find the angles that correspond to those ratios.

5-24. Mr. Gow needs to build a wheelchair access ramp for the school's auditorium. The ramp must rise a total of 3 feet to get from the ground to the entrance of the building. In order to follow the state building code, the angle formed by the ramp and the ground cannot exceed 4.76°.

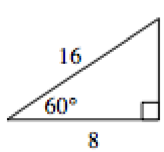
Mr. Gow has plans from the planning department that call for the ramp to start 25 feet away from the building. Will this ramp meet the state building code?

a. Draw an appropriate diagram. Add all the measurements you can. What does Mr. Gow need to find?

b. To find an angle from a trigonometric ratio you need to "undo" it, just like you can undo addition with subtraction, multiplication with division, or squaring by finding the square root. These examples are all pairs of **inverse** operations.

When you use a calculator to do this, you use inverse trigonometric functions which are usually labeled "**sin**$^{-1}$", "**cos**$^{-1}$", and "**tan**$^{-1}$". These are pronounced, "inverse sine," "inverse cosine," and "inverse tangent." On many calculators, you must press the "inv" or "2nd" key first, then the "sin", "cos", or "tan" key.

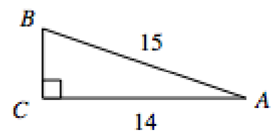
Verify that your calculator can find an inverse trig value using the triangle below from Lesson 5.1.2. When you find $\cos^{-1}\frac{8}{16}$, do you get 60°?



c. Return to your diagram from part (a). According to the plan, what angle will the ramp make with the ground? Will the ramp be to code?

d. At least how far from the building must the ramp start in order to meet the building code? If Mr. Gow builds the ramp exactly to code, how long will the ramp be? Show all work.

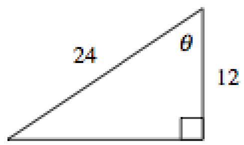
5-25. For the triangle at right, find the measures of $\angle A$ and $\angle B$. Once you have found the measure of the first acute angle (either $\angle A$ or $\angle B$), what knowledge about the angles in triangles could help you find the second acute angle?

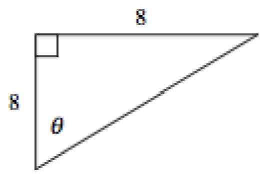


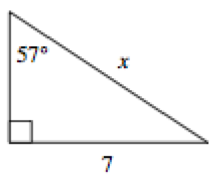
5-26. Examine the triangles below. Note: The diagrams are not drawn to scale.

With your study team:

- Look through all the triangles first and see if any look familiar or are ones that you know how to answer right away without using a trigonometric tool.
- Then, for all the other triangles, identify which tool to use based on where the reference angle (q) is located and which side lengths are involved.
- Write and solve an equation to find the missing side length or angle.

a. 

b. 

c. 

5-27. Peter cannot figure out what he did wrong. He wrote the equation below to find the missing angle of a triangle. However, his calculator gives him an error message each time he tries to calculate the angle.

Peter's work: $\cos\theta = \frac{8}{2.736}$

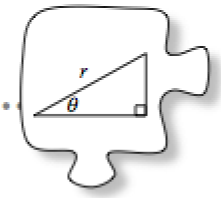
Jeri, his teammate, looked at his work and exclaimed, "*Of course your calculator gives you an error! Your cosine ratio is impossible!*" What is Jeri talking about? And how could she tell without seeing the triangle or checking on her calculator?

5-28. LEARNING LOG

Write a Learning Log entry describing what you know about inverse trig functions. Be sure to include an example and a description of how to solve it. Title this entry, “Inverse Trig Functions” and label it with today’s date.

5.1.4 How can I use trig ratios?

Trigonometric Applications



Throughout this chapter, you have developed new tools to help you determine the length of any side or the measurement of any angle of a right triangle. Trigonometric ratios, coupled with the Pythagorean Theorem, give you the powerful ability to solve problems involving right triangles. Today you will apply this knowledge to solve some real world problem situations.

As you are working with your team on the problems below, be sure to draw and label a diagram and determine which trigonometric ratio to use before you start solving.

5-36. CLIMBING IN YOSEMITE

David and Emily are climbing El Capitan, a big cliff wall in Yosemite National Park. David is on the ground holding the rope attached to a carabineer (a rope "pulley" that is on the wall) above Emily as she climbs. When Emily stops to rest, David wonders how high she has climbed. The rope is attached to his waist, about 3 feet off of the ground, and he has let out 48 feet of rope which goes up to the carabineer and then back down the wall to Emily's harness. The rope at David's waist makes a 55° angle with the ground and he is standing 20 feet away from the base of the wall.

a. Assuming that the rope is taut (i.e., pulled tight), approximately how long is the rope between David and the carabineer above Emily?

b. How high up the wall has Emily climbed? Describe your method.

5-37. The Bungling Brothers Circus is in town and you are part of the crew that is setting up its enormous tent. The center pole that holds up the tent is 70 feet tall. To keep it upright, a support cable needs to be attached to the top of the pole so that the cable forms a 60° angle with the ground.

a. How long is the cable?

b. How far from the pole should the cable be attached to the ground?

5-38. Nathan is standing in a meadow, exactly 185 feet from the base of El Capitan. At 11:00 a.m., he observes Emily climbing up the wall, and determines that his angle of sight up to Emily is about 10°. Use the 5-38 Student eTool (Desmos) to visualize the problem.

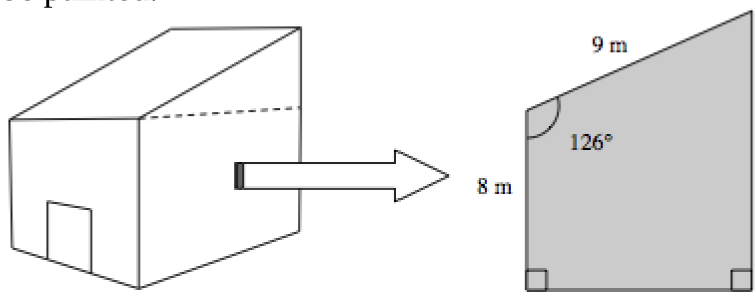
a. If Nathan's eyes are about 6 feet above the ground, about how high is Emily at 11:00 a.m.?

b. At 11:30 a.m., Emily has climbed some more, and Nathan's angle of sight to her is now 25°. How far has Emily climbed in the past 30 minutes?

c. If Emily climbs 32 feet higher in ten more minutes, at what angle will Nathan have to look in order to see Emily?

5-39. Forest needs to repaint the right side of his house because sunlight and rain have caused the paint to peel. Each can of paint states that it will cover 150 sq. feet.
Help Forest decide how many cans of paint he should buy.

a. Copy the shaded diagram below onto your paper. Work with your team to find the area that will be painted.



b. Assuming that Forest can only buy whole cans of paint, how many cans of paint should he buy? (Note: 1 square meter ≈ 10.764 square feet)

5-40. TEAM CHALLENGE

It is 11:55 a.m., and Emily has climbed even higher. The rope now makes an angle of 6° with the cliff wall. If David is 18 feet away from the base of El Capitan, at what angle should Nathan (who is 185 feet from the base) look up to see Emily?

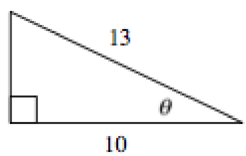
METHODS AND MEANINGS

MATH NOTES

Inverse Trigonometry

Just as subtraction "undoes" addition and multiplication "undoes" division, the inverse trigonometric functions "undo" the trigonometric functions tangent, sine, and cosine. Specifically, inverse trigonometric functions (**inverse cosine, inverse sine, and inverse tangent**) are used to find the measure of an acute angle in a right triangle when a ratio of two sides is known. This is the **inverse**, or opposite, of finding the trigonometric ratio from a known angle.

The inverse trigonometric functions that will be used in this course are: $\mathbf{sin^{-1}}$, $\mathbf{cos^{-1}}$, and $\mathbf{tan^{-1}}$ (pronounced "inverse sine," "inverse cosine" and "inverse tangent"). Below is an example that shows how $\cos^{-1}$ may be used to find a missing angle, θ.



$$\cos\theta = \frac{10}{13}$$
$$\theta = \cos^{-1}\left(\frac{10}{13}\right)$$
$$\theta \approx 39.7°$$

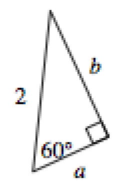
To evaluate $\cos^{-1}\left(\frac{10}{13}\right)$ on a scientific calculator, most calculators require the "2nd" or "INV" button to be pressed before the "cos" button.

5.2.1 Is there a shortcut?

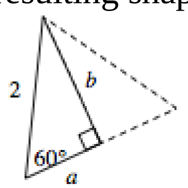
Special Right Triangles

You know when triangles are similar and how to find missing side lengths in similar triangles. Today you will be using both of those ideas to investigate patterns within two types of special right triangles. These patterns will allow you to use a shortcut whenever you need to find side lengths in these particular types of right triangles.

5-47. Darren wants to find the side lengths of the triangle below. The only problem is that he left his calculator at home and he does not remember the value of 60°.



a. "*That's okay,*" says his teammate, Jan. "*I think I see a shortcut*." Using tracing paper, she created the diagram below by reflecting the triangle across the side with length *b*. What is the resulting shape? How do you know?

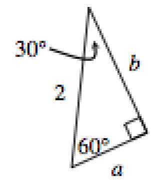


b. What if Jan had reflected across a different side? Would the result still be an equilateral triangle? Why or why not?

c. Use Jan's diagram to find the value of *a* without using a trigonometric ratio.

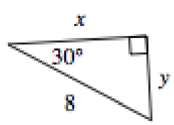
d. Now find the length of *b* without a calculator. Leave your answer in exact form. In other words, do not approximate the height with a decimal.

5-48. Darren's triangle is an example of a half-equilateral triangle, also known as a 30°- 60°- 90° triangle because of its angle measures. Darren is starting to understand Jan's shortcut, but he still has some questions. Help Darren by answering his questions below.



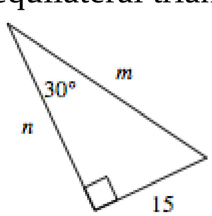
a. "*Will this approach work on all triangles?*" In other words, can you always form an equilateral triangle by reflecting a right triangle? Explain your reasoning.

b. "*What if the triangle is a different size?*" Use tracing paper to show how Darren can reflect the triangle below to form an equilateral triangle. Then find the lengths of *x* and *y* without a calculator.

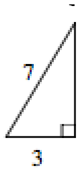


c. "*Is the longer leg of a 30°- 60°- 90° triangle always going to be the length of the shorter leg multiplied by* $\sqrt{3}$ *?*" Explain why or why not.

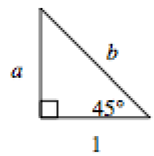
d. "*What if I only know the length of the shorter leg?*" Consider the triangle below. Visualize the equilateral triangle. Then find the values of *n* and *m* without using a calculator.



e. Darren drew the triangle below and is wondering if it also is a 30°- 60°- 90° triangle. What do ink? How do you know?



5-49. Darren wonders if he can find a similar pattern in another special triangle he knows, shown below.

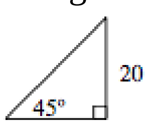


a. Use what you know about this triangle to help Darren find the lengths of *a* and *b* without a trig tool or a calculator. Leave your answer in exact form.

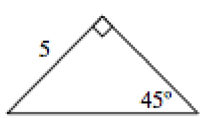
b. What should Darren name this triangle?

c. Use the fact that all 45°- 45°- 90° triangles are similar to find the missing side lengths in the right triangles below. Leave your answers in exact, radical form.

(1)

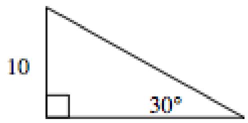


(2)

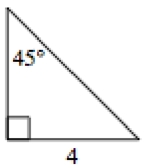


5-50. Use your new 30°- 60°- 90° and 45°- 45°- 90° triangle patterns to quickly find the lengths of the missing sides in each of the triangles below. Do not use a calculator. Leave answers in exact form. Note: The triangles are not necessarily drawn to scale.

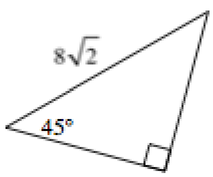
a.



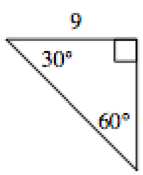
b.



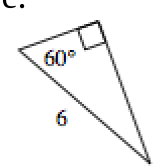
c.



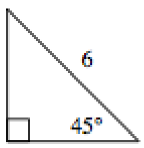
d.



e.



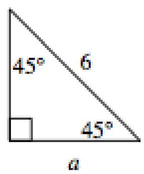
f.



5-51. LEARNING LOG

In your Learning Log, explain what you know about 30°- 60°- 90° and 45°- 45°- 90° triangles. Include diagrams of each. Label this entry “Special Right Triangles” and label it with today’s date.

Rationalizing a Denominator



In Lesson 5.2.1, you developed some shortcuts to help find the lengths of the sides of a 30°- 60°- 90° and 45°- 45°- 90° triangle. This will enable you to solve similar problems in the future without a calculator or your Trig Table Toolkit.

However, sometimes using the shortcuts leads to some strange looking answers. For example, when finding the length of *a* in the triangle at right, you will get the expression $\frac{6}{\sqrt{2}}$.

A number with a radical in the denominator is difficult to estimate. Therefore, it is sometimes beneficial to **rationalize the denominator** so that no radical remains in the denominator. Study the example below.

Example: Simplify $\frac{6}{\sqrt{2}}$.

First, multiply the numerator and denominator by the radical in the denominator. Since $\frac{\sqrt{2}}{\sqrt{2}} = 1$, this does not change the value of the expression.

<u>Example</u>

$$\frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2}$$
$$= 3\sqrt{2}$$

After multiplying, notice that the denominator no long has a radical, since $\sqrt{2} \cdot \sqrt{2} = 2$.

Often, the product can be further simplified. Since 2 divides evenly into 6, the expression $\frac{6\sqrt{2}}{2}$ can be rewritten as $3\sqrt{2}$.

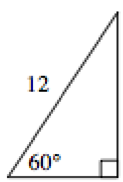
5.2.2 How can I use similar triangles?

Pythagorean Triples

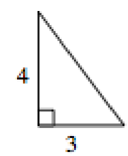
In Lesson 5.2.1, you developed shortcuts that will help you quickly find the lengths of the sides of certain right triangles. What other shortcuts can be helpful? As you work today with your study team, look for patterns and connections between triangles.

5-59. Use the tools you have developed thus far to find the lengths of the missing sides of the triangles below. If you know of a shortcut, share it with your team. Be ready to share your strategies with the class.

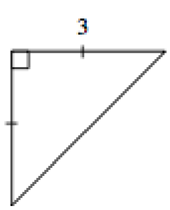
a.



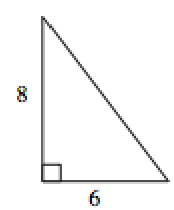
c.



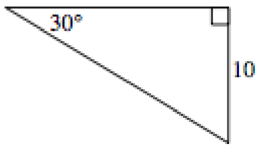
b.



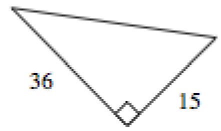
d.



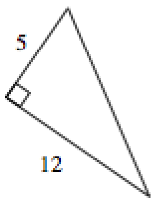
e.



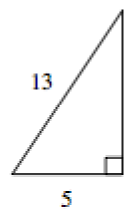
h.



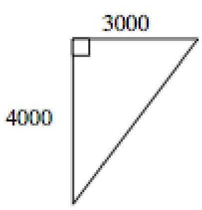
f.



i.



g.



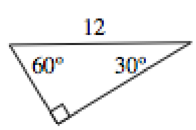
5-60. Karl noticed some patterns as he was finding lengths of the missing sides of the triangles in problem 5-59. He recognized that the side lengths of the triangles in parts (c), (d), (f), (g), (h), and (i) were integers. He also noticed that knowing the side lengths of the triangle in part (c) could help find the length of the hypotenuse in parts (d) and (g).

a. Groups of numbers like 3, 4, 5 and 5, 12, 13 are called **Pythagorean Triples**. Why do you think they are called Pythagorean Triples?

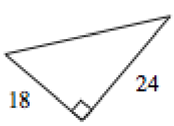
b. What other sets of numbers are also Pythagorean Triples? How many different sets can you find?

5-61. With your team, find the missing side lengths for each triangle below. Try to use your new shortcuts whenever possible.

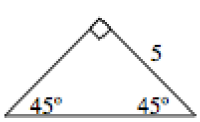
a.



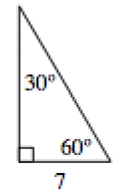
b.



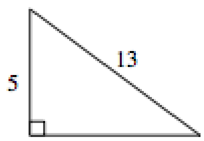
c.



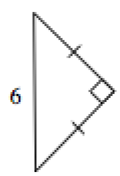
d.



e.

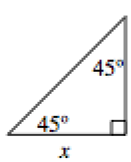


f.

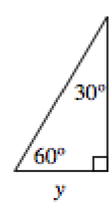


5-62. Diana looked at the next problems and thought she could not do them. Erik pointed out that they are the same as the previous triangles, but instead of numbers, each side length is given in terms of a variable. Use Erik's idea to help you find the missing side lengths of the triangles below:

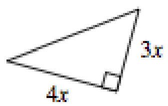
a.



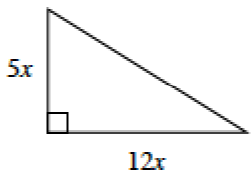
b.



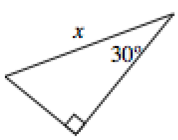
c.



d.

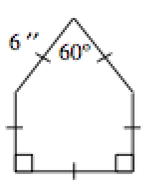


e.

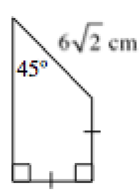


5-63. Use your shortcuts to find the area and perimeter of each shape below:

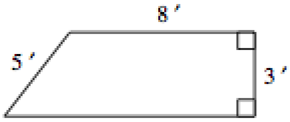
a.



b.



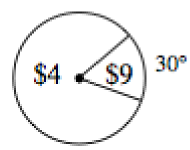
c.



LOOKING DEEPER MATH NOTES

Expected Value

The amount you would expect to win (or lose) per game after playing a game of chance many times is called the **expected value**. This value does not need to be a possible outcome of a single game, but instead reflects an average amount that will be won or lost per game.



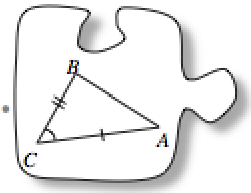
For example, the "$9" portion of the spinner at right makes up $\frac{30}{360}=\frac{1}{12}$ of the spinner, while the "$4" portion is the rest, or $\frac{11}{12}$, of the spinner. If the spinner was spun 12 times, probability predicts that it would land on "$9" once and "$4" eleven times. Therefore, someone spinning 12 times would expect to receive 1($9) + 11($4) = $53. On average, each spin would earn an expected value of $\frac{\$53}{12\text{ spins}} \approx \$4.42$. You could use this value to predict the result for any number of spins. For example, if you play 30 times, you would expect to win 30($4.42) = $132.50.

Another way to calculate expected value involves the probability of each possible outcome. Since "$9" is expected $\frac{1}{12}$ of the time, and "$4" is expected $\frac{11}{12}$ of the time, then the expected value can be calculated with the expression $(\$9)(\frac{1}{12}) + (\$4) + (\frac{11}{12}) = \frac{\$53}{12} \approx \$4.42$.

A **fair game** is one in which the expected value is zero. Neither player expects to win or lose if the game is played numerous times.

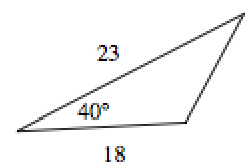
5.3.1 What triangle tools do I still need?

Finding Missing Parts of Triangles



When do you have enough information to find all of the angle measures and side lengths of a triangle? For example, can you find all of the side lengths if you are only informed about the three angles? Does it matter if the triangle has a right angle or not? Today you will organize your triangle knowledge so that you know what tools you have and for which triangles you can accurately find all of the angle measures and side lengths.

5-71. How many ways can three pieces of information about a triangle be given? For example, the three given measurements could be one angle and two side lengths, as shown in the triangle at right. List as many other combinations of three pieces of information about a triangle as you can.



5-72. HOW MUCH INFORMATION DO YOU NEED?

So far in this course you have developed several tools to find missing parts of triangles. But how complete is your Triangle Toolkit? Are there more tools that you need to develop?

Your Task: With your team, find the missing angles and sides of the triangles below. (Also printed on the Lesson 5.3.1 Resource Page). Notice that each triangle has three given pieces of information about its angles and sides. If there is not enough information or if you do not yet have the tools to find the missing information, explain why.

Discussion Points

- Are there any triangles that look familiar or that you already have a strategy for?
- What tools do you have to solve for parts of triangles? For what types of triangles do these tools work?
- Would it be helpful to subdivide any of the triangles into right triangles?

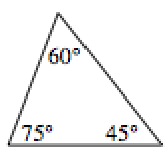
Further Guidance

5-73. While looking for different strategies to use, advice from a teammate can often help.

a. Angelo thinks that the Pythagorean Theorem is a useful tool. For which types of triangles is the Pythagorean Theorem useful? Look for these types of triangles in problem 5-72 and use the theorem to solve for any missing sides.

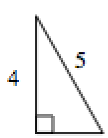
b. Tomas remembers using trigonometric ratios to find the missing sides and angles of a triangle. Which triangles from problem 5-72 can be analyzed using this strategy?

c. Ngoc thinks that more than one triangle exists with the angles at right. Is she correct? If so, how are these triangles related?

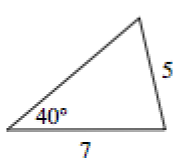


Three angles

d. Does it matter if the triangle is a right triangle? For example, both of these triangles (from parts (b) and (d)) give an angle and two sides. Can you use the same tool for both? Why or why not?



A right angle and two sides



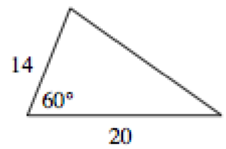
Two sides and an angle not between them

Further Guidance section ends here.

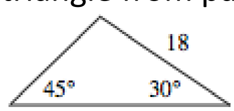
5-74. WHAT IF IT DOES NOT HAVE A RIGHT ANGLE?

If your team needs help on parts (c) and (f) of problem 5-72, Leila has an idea. She knows that she has some tools to use with right triangles but noticed that some of the triangles in problem 5-72 are *not* right triangles. Therefore, she thinks it is a good idea to split a triangle into two right triangles.

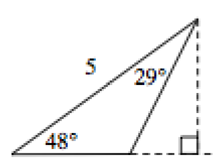
a. Discuss with your team how to change the diagram below so that the triangle is divided into two right triangles. Then use your right triangle tools to solve for the missing sides and angles.



b. Leila wonders if her method would work for other triangles too. Test her method on the triangle from part (f) of problem 5-62 (also shown below). Does her method work?



5-75. Ryan liked Leila's idea so much that he looked for a way to create a right triangle in the triangle from part (g) of problem 5-72. He decided to draw a height *outside* the triangle, forming a large right triangle. Use the right triangle to help you find the missing side lengths of the original triangle.



5-76. LEARNING LOG

Return to problem 5-72 and examine all of the ways that three pieces of information can be given about a triangle. For which triangle(s) were you able to find missing side lengths and angles? For which triangle(s) do you not have enough information given? For which triangle(s) do you need a new strategy? Reflect on the strategies you have developed so far. Title this entry “Strategies to find Sides and Angles of a Triangle” and label it with today’s date.

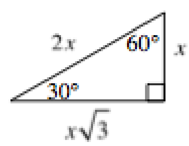
METHODS AND MEANINGS

MATH NOTES

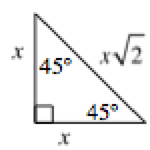
Special Right Triangles

So far in this chapter, you have learned about several special right triangles. Being able to recognize these triangles will enable you to quickly find the lengths of the sides and will save you time and effort in the future.

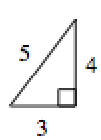
The half-equilateral triangle is also known as the **30°- 60°- 90° triangle**. The sides of this triangle are always in the ratio $1:\sqrt{3}:2$, as shown below.

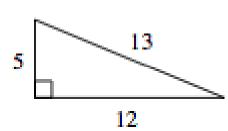


Another special triangle is the **45°- 45°- 90° triangle**. This triangle is also commonly known as an isosceles right triangle. The ratio of the sides of this triangle is always $1:1:\sqrt{2}$.



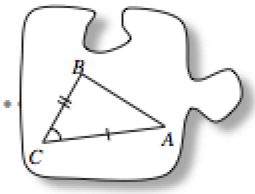
You also discovered several **Pythagorean Triples**. A Pythagorean Triple is any set of 3 positive integers *a*, *b*, and *c* for which $a^2 + b^2 = c^2$. Two of the common Pythagorean Triples that you will see throughout this course are shown below.





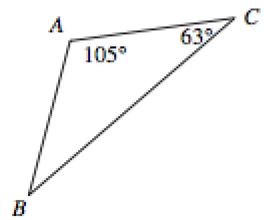
5.3.2 Is there a faster way?

Law of Sines

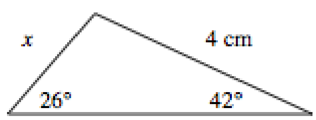


In problem 5-74, you used a complicated strategy to find the lengths of sides and measures of angles for a non-right triangle. Is there a tool you can use to find angles and side lengths of non-right triangles directly, using fewer steps? Today you will explore the relationships that exist among the sides and angles of triangles and will develop a new tool called the Law of Sines.

5-83. Is there a relationship between a triangle's side and the angle opposite to it? For example, assume that the diagram for ΔABC, shown below, is not drawn to scale. Based on the angle measures provided in the diagram, which side must be longest? Which side must be shortest? How do you know?



5-84. When Madelyn examined the triangle below, she said, "*I don't think this diagram is drawn to scale because I think the side labeled x has to be longer than 4 cm.*"

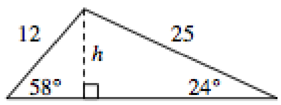


a. Do you agree with Madelyn? Why or why not?

b. Leila thinks that x can be found by using right triangles. Review what you learned in Lesson 5.3.1 by finding the value of x.

c. Find the area of the triangle.

5-85. Thui and Ivan came up with two different ways to find the height of the triangle below.



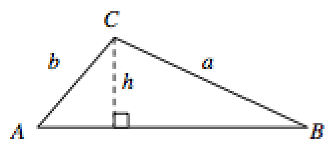
- Using the right triangle on the left, Thui wrote: $\sin 58° = \frac{h}{12}$.
- Ivan also used the sine function, but his equation looked like this: $\sin 24° = \frac{h}{25}$.

a. Which triangle did Ivan use?

b. Calculate h using Thui's equation and again using Ivan's equation. How do their answers compare?

5-86. LAW OF SINES

Edwin wonders if Thui's and Ivan's methods can be used to find a way to relate the sides and angles of a non-right triangle. To find the height, Ivan and Thui drew a perpendicular line from point C to side $\overline{AB}$. Then each used the sine ratio with an acute angle and the hypotenuse of a right triangle.

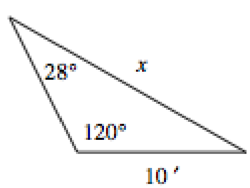


a. Use the triangle above to find two expressions for h using the individual right triangles like you did in problem 5-85.

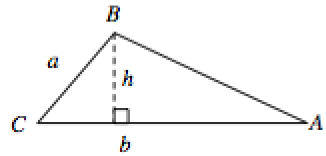
b. Use your expressions from part (a) to show that $\frac{\sin(m\measuredangle B)}{b} = \frac{\sin(m\measuredangle A)}{a}$.

c. Describe where $\angle B$ is located in relation to the side labeled b. How is $\angle A$ related to the side labeled a?

d. The relationship $\frac{\sin(m\measuredangle B)}{b} = \frac{\sin(m\measuredangle A)}{a}$ is called the **Law of Sines**. Read the Math Notes box for this lesson to learn more about this relationship. Then use this relationship to solve for $x$in the triangle below.



5-87. The sine ratio can also help to find the area of a triangle when any two side lengths and the measure of the included angle is given. Explore this fact below.



a. Assume for the triangle above that you know the lengths of a and b and the measure of the angle at C. Also assume that h is the height of the triangle found by dropping a perpendicular from B to its opposite side. Write an expression for the area of this triangle using its base and height.

b. What if the height is unknown? Since $m\angle C$ is known, write an expression for the $\sin C$ using one of the right triangles. Solve for h.

c. Use your expression for h in part (b) to rewrite the area expression you wrote for part (a). You now have a relationship that you can use to find the area of any triangle when you know two sides lengths and the measure of the angle between them.

d. Use your relationship from part (c) to find the area of the triangle in part (d) of problem 5-86. Remember that you will need the measure of the angle between the two side lengths.

5-88. LEARNING LOG

Reflect on what you have learned during this lesson about the sides and angles of a triangle. What is the Law of Sines and when can it be used? Include an example. Title this entry "Law of Sines" and label it with today's date.

METHODS AND MEANINGS

MATH NOTES

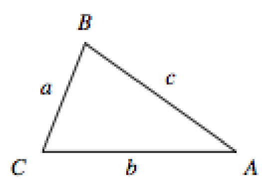
Law of Sines

For any ΔABC , the ratio of the sine of an angle to the length of the side opposite the angle is constant. This means that:

$$\frac{\sin(m\angle A)}{a} = \frac{\sin(m\angle B)}{b},$$

$$\frac{\sin(m\angle B)}{b} = \frac{\sin(m\angle C)}{c}, \text{ and}$$

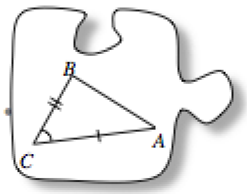
$$\frac{\sin(m\angle A)}{a} = \frac{\sin(m\angle C)}{c}.$$



This property is called the **Law of Sines**. This is a powerful tool because you can use the sine ratio to solve for measures of angles and lengths of sides of *any* triangle, not just right triangles. The law works for angle measures between 0° and 180°.

5.3.3 How can I complete my triangle toolkit?

Law of Cosines

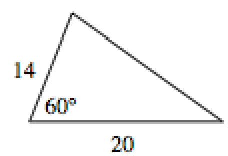


So far, you have three tools that will help you solve for missing sides and lengths of a triangle. In fact, one of those tools, the Law of Sines, even helps when the triangle is not a right triangle. There are still two triangles from our exploration in Lesson 5.3.1 that you cannot solve directly with any of your existing tools. Today you will develop a tool to help find missing side lengths and angle measures for triangles such as those shown at right.

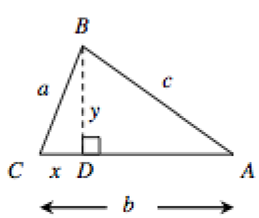
By the end of this lesson, you will have a complete set of tools to help solve for the side lengths and angle measures of *any* triangle, as long as enough information is given.

5-96. LAW OF COSINES

Leila remembers that in problem 5-74, she solved for the side lengths and missing angles of the triangle at right by dividing the triangle into two right triangles. She thinks that using two right triangles may help find a tool that works for any triangle with two given sides and a given angle between them. Help Leila generalize this process by answering the questions below.



a. Examine ΔABC below. Assume that you know the lengths of sides a and b and the measure of $\angle C$. Notice how the side opposite $\angle A$ is labeled a and the side opposite $\angle B$ is labeled b, and so on. Line segment $\overline{BD}$ is drawn so that ΔABC is divided into two right triangles. If $CD = x$, then what is DA?



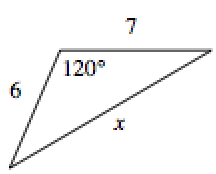
b. Write an equation relating a, x, and y. Likewise, write an equation relating the side lengths of ΔBDA.

c. Leila noticed that both expressions from part (b) have a y^2-term. "*Can we combine these equations so that we have one equation that links sides a, b, and c?*" she asked. With your team, use algebra to combine these two equations so that y^2 is eliminated. Then simplify the resulting equation as much as possible.

d. The equation from part (c) still has an x-term. Using only a and $m\angle C$, find an expression for x using the left-hand triangle. Solve your equation for x, then substitute this expression into your equation from part (c) for x.

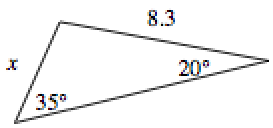
e. Solve your equation from part (d) for c^2. You have now found an equation that links the lengths of two sides and the measure of the angle between them to find the length of the side opposite the angle. This relationship is called the **Law of Cosines**. Read the Math Notes box for this lesson to learn more about the Law of Cosines.

f. Use the Law of Cosines to solve for x in the triangle below.

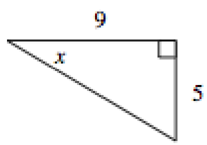


5-97. You now have several tools to solve for missing side lengths and angle measures. Decide which tool to use for each of the triangles below and solve for *x*. Decide if your answer is reasonable based on the diagram.

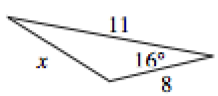
a.



b.

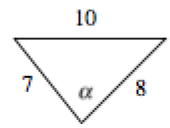


c.



5-98. EXTENSION

Not only can the Law of Cosines be used to solve for side lengths, but it can also be used to solve for angles. Consider the triangle from Lesson 5.3.1, shown below.



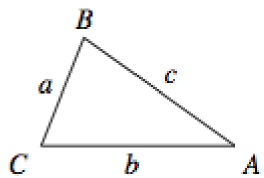
a. Write an equation that relates the three sides and the angle *a*. Then solve the equation for α.

b. Now solve for the other two angles using any method. Be sure to name which tool(s) you used!

5-99. You have now completed your Triangle Toolkit and can find the missing side lengths and angle measures for *any* triangle, provided that enough information is given. Add the Law of Sines and Law of Cosines to your Triangle Toolkit for reference later in this course.

Methods and Meanings
Math Notes

Law of Cosines



Just like the Law of Sines, the **Law of Cosines** represents a relationship between the sides and angles of a triangle.

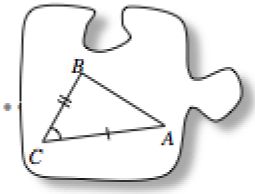
Specifically, when given the lengths of any two sides, such as a and b, and the angle between them, $\angle C$, the length of the third side, in this case c, can be found using this relationship

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Similar equations can be used to solve for a and b.

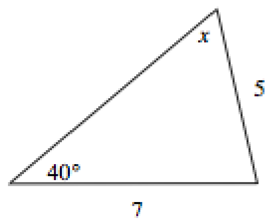
5.3.4 Is there more than one possible triangle?

Ambiguous Triangles



Now that you have completed your Triangle Toolkit, you can solve for the missing angles or sides of any triangle, provided that enough information is given. But how do you know if you have enough information? What if there is more than one possible triangle? Today you will explore situations where your tools may not be adequate to solve for the missing side lengths and angle measures of a triangle.

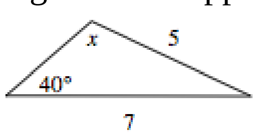
5-106. Examine the triangle below from part (d) of problem 5-72. Notice that two side lengths and an angle measure not between the labeled sides are given. (This situation is sometimes referred to as SSA.)



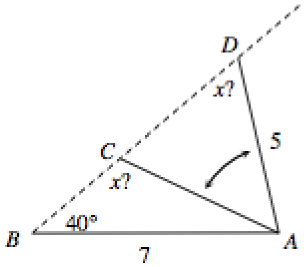
a. Assuming that the triangle is not drawn to scale, what do you know about x? Is it more than 40° or less than 40°? Justify your conclusion.

b. Solve for x. Was your conclusion from part (a) correct?

c. "*Hold on!*" proclaims your teammate, Missy. "*That's not what I got. I found out that $x \approx 115.9°$.*" She then drew the triangle below. Do you agree with Missy? Use the Law of Sines to test her answer. That is, find out if the ratios of $\frac{\sin(\text{angle})}{\text{opposite side}}$ are equal for each angle and its opposite side.



d. What happened? How can there be two possible answers for x? Examine the diagram below for clues.



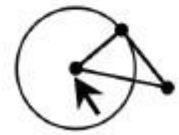
e. What is the relationship between the two solution angles? Do you think this relationship always exists? Examine the diagram above, which shows the two angle solutions, and use it to explain how the solutions are related.

5-107. In problem 5-106, you determined that it was possible to create two different triangles because you were given only two side lengths and an angle not between them. When this happens, it is called **triangle ambiguity** since you cannot tell which triangle was the one you were supposed to find. Will there always be two possible triangles? Can there ever be more than two possible triangles? Think about this as you answer the questions below.

a. Use the or obtain the Lesson 5.3.4 Resource Page and some linguini (or other flat manipulative) from your teacher. Prepare pieces of linguini that are 1 inch, 1.5 inches, 2 inches, 2.5 inches, and 3 inches long.

b. For each length of linguini, place one end at point A in the diagram on the resource page. Determine if you can form a triangle by connecting the linguini with the dashed side to close the triangle. If you can make a triangle, label the third vertex C and label $\overline{AC}$ with its length. Can you form more than one triangle with the same side length? Is a triangle always possible? Record any conjectures you make.

c. If the technology is available, test your conjectures from part (b) with a dynamic geometry tool. Try to learn everything you can about SSA triangles. Use the questions below to guide your investigation.

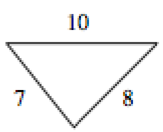


- Can you find a way to create three possible triangles with one set of SSA information?
- Is it ever impossible to form a triangle?
- Is it possible to choose SSA information that will create only one triangle? How?

5-108. Now that Alex knows that SSA (two sides and an angle not between them) can result in more than one possible triangle, he wants to know if other types of given information can also create ambiguous results. For example, when given three side lengths, is more than one triangle possible?

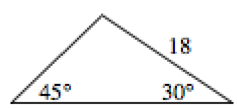
Examine each of the diagrams below (from problem 5-72) and determine if any other types of triangles are also ambiguous. You may want to imagine building the shapes with linguini. Remember that the given information cannot change – thus, if a side length is given, it cannot be lengthened or shortened.

a.



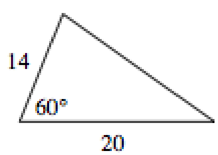
Three sides (SSS)

c.



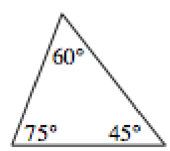
Two angles and a side not between them (AAS)

b.



Two sides and an angle between them (SAS)

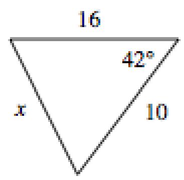
d.



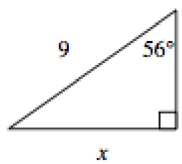
Three angles (AAA)

5-109. You now have several tools to use when solving for missing sides lengths and angle measures. Decide which tool to use for each of the triangles below and solve for *x*. If there is more than one solution, find both. Name the tool you use.

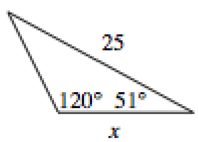
a.



b.

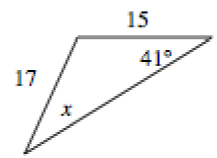


c.



5-110. EXTENSION

While examining the triangle below, Alex stated, "*Well, I know that there can be at most one solution.*"

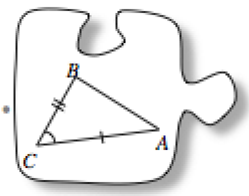


a. Examine the information given in the triangle. What do you know about x? Is it more than 41° or less? How can you tell?

b. Alex remembered that if there were two solutions, then they had to be supplementary. Explain why this triangle cannot have two different values for x.

5.3.5 Which tool should I use?

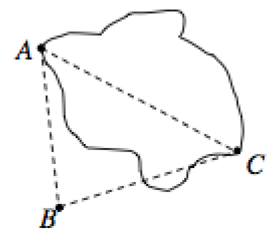
Choosing a Tool



During this section, you have developed new tools such as the Law of Sines and the Law of Cosines to find the lengths of sides and the measures of angles of a triangle. These strategies are very useful because they work with all triangles, not just right triangles. But when is each strategy the best one to use? Today you will focus on which strategy is most effective to use in different situations. You will also apply your strategies to triangles in different contexts.

As you work on these problems, keep in mind that good communication and a joint brainstorming of ideas will greatly enhance your team's ability to choose a strategy and to solve these problems.

5-118. LAKE TOFTEE, Part One

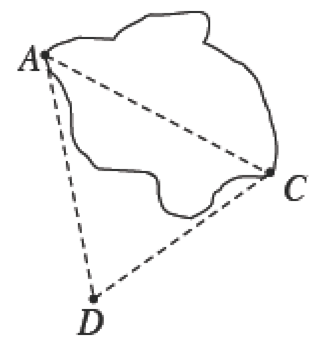


A bridge is being designed to connect two towns along the shores of Lake Toftee in Minnesota (one at point A and the other at point C). Lavanne has been given the responsibility of determining the length of the bridge.

Since he could not accurately measure across the lake (AC), he measured the only distance he could by foot (AB). He drove a stake into the ground at point B and found that $AB = 684$ feet. He also used a protractor to determine that $m\angle B = 79°$ and $m\angle C = 53°$. How long will the bridge need to be?

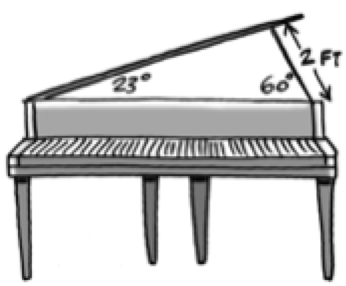
5-119. LAKE TOFTEE, Part Two

Lavanne was not convinced that his measurements from problem 5-118 were correct. He decided to measure the distance between towns *A* and *C* again using a different method to verify his results.



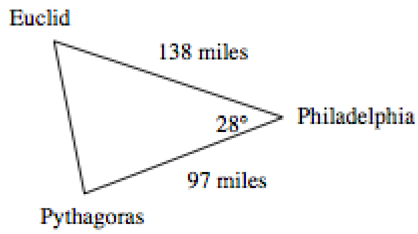
This time, he decided to drive a stake in the ground at point *D*, which is 800 feet from town *A* and 694 feet from town C. He also determined that $m\angle D = 68^{\circ}$. Using these measurements, how wide is the lake between points *A* and *C*? Does this confirm the results from problem 5-106?

5-120. The lid of a grand piano is propped open by a supporting arm, as shown in the diagram below. Carson knows that the supporting arm is 2 feet long and makes a 60° angle with the piano. He also knows that the piano lid makes a 23° angle with the piano. How wide is the piano?



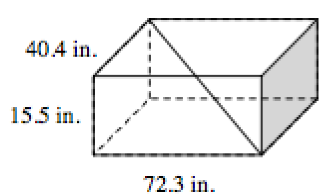
5-121. PENNANT RACE

Your basketball team has made it to the semi-finals and now needs to win only two more games to go to the finals. Your plan is to leave Philadelphia, travel 138 miles to the town of Euclid to play the team there. Then you will leave Euclid, travel to Pythagoras, and play that team. Finally, you will travel 97 miles to return home.

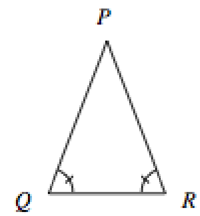


Your team bus can travel only 300 miles on one tank of gas. Assuming that all of the roads connecting the three towns are straight and that the two roads that connect in Philadelphia form a 28° angle, will one tank of gas be enough for the trip? Justify your solution.

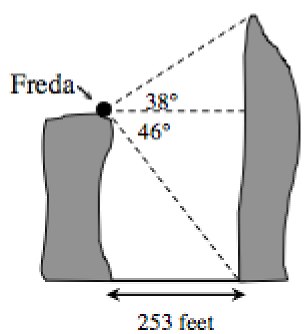
5-122. Shonte is buying pipes to install sprinklers for her front lawn. She needs to fit the pipes into the bed of her pickup truck to get them home from the store. She knows that the longest dimension in her truckbed is from the top of one corner to the bottom of the bed at the opposite corner. After measuring the truckbed, she drew the diagram below. What is the longest pipe she can buy? View the virtual 3D Model Box (CPM).



5-123. You have previously used your intuition to state that if a triangle has two angles that are equal, then the triangle has two sides that are the same length. Now use your triangle tools to show that your intuition was correct. For example, for ΔPQR, show that if $m\angle Q = m\angle R$, then $PQ = PR$.



5-124. As Freda gazes at the edge of the Grand Canyon, she decides to try to determine the height of the wall opposite her. Using her trusty clinometer, she determines that the top of the wall is at a 38° angle above her, while the bottom is at a 46° angle below her, as shown in the diagram below. If the base of the wall is 253 feet from the point on the ground directly below Freda, determine the height of the wall opposite her.



5-125. While facing north, Lisa and Aaron decide to hike to their campsite. Lisa plans to hike 5 miles due north to Lake Toftee before she goes to the campsite. Aaron plans to turn 38° east and hike 7.4 miles directly to the campsite. How far will Lisa have to hike from the lake to meet Aaron at the campsite? Start by drawing a diagram of the situation, then calculate the distance.