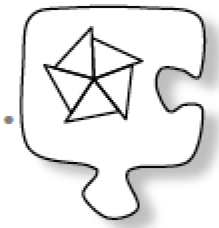
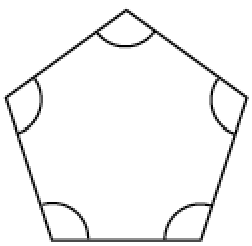
# 8.1.2 What is its measure?



## Interior Angles of Polygons

In an earlier chapter you discovered that the sum of the interior angles of a triangle is always 180°. What about other polygons, such as hexagons or decagons? What about the sum of their interior angles? Do you think it matters if the polygon is convex or not? Consider these questions today as you investigate the angles of a polygon.

**8-13.** Copy the diagram of the regular pentagon below onto your paper. Then, with your team, find the *sum* of the measures of its interior angles *as many ways as you can*. You may want to use the fact that the sum of the angles of a triangle is 180°. Be prepared to share your team's methods with the class.



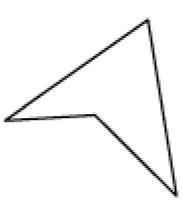
**8-14.** SUM OF THE INTERIOR ANGLES OF A POLYGON

In problem 8-13, you found the sum of the angles of a regular pentagon. But what about other polygons?

a. Obtain a Lesson 8.1.2 Resource Page from your teacher. Then use one of the methods from problem 8-13 to find the sum of the interior angles of other polygons. Complete the table (also shown below) on the resource page.

| Number of Sides of the Polygon | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 |
|---|---|---|---|---|---|---|---|---|---|
| Sum of the Interior Angles of the Polygon | 180° | | | | | | | | |

b. Does the interior angle sum depend on whether the polygon is convex? Test this idea by drawing a few non-convex polygons (like the one below) on your paper and determine if it matters whether the polygon is convex. Explain your findings.

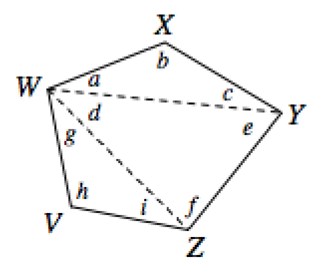


c. Find the sum of the interior angles of a 100-gon. Explain your reasoning.

d. LEARNING LOG

In your Learning Log, write an expression that represents the sum of the interior angles of an *n*-gon. Title this entry "Sum of Interior Angles of a Polygon" and include today's date.

**8-15.** The pentagon at right has been dissected (broken up) into three triangles with the angles labeled as shown. Use the three triangles to prove that the sum of the interior angles of *any* pentagon is always 540°. If you need help doing this, answer the questions below.

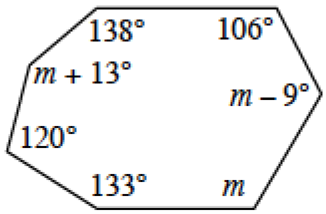


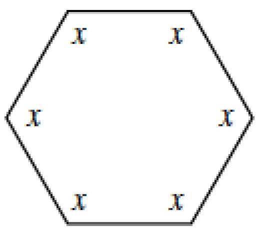

a. What is the sum of the angles of a triangle? Use this fact to write three equations based on the triangles in the diagram.

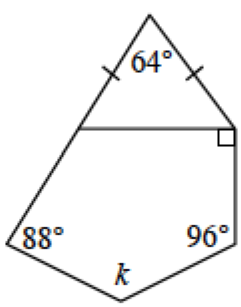
b. Add the three equations to create one long equation that represents the sum of all nine angles.

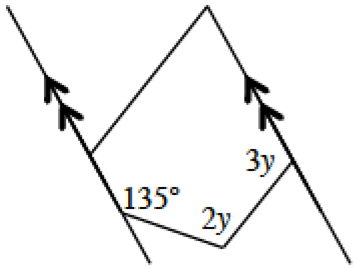
c. Substitute the three-letter name for each angle of the pentagon for the lower case letters at each vertex of the pentagon. For example, $\angle XYZ = c + e$.

**8-16.** Use the angle relationships in each of the diagrams below to solve for the given variables. Show all work.

a. 


b. 


c. 

d. 


# 8.1.3 What if it is a regular polygon?

## Angles of Regular Polygons

In Lesson 8.1.2 you discovered how to determine the sum of the interior angles of a polygon with any number of sides. What more can you learn about a polygon? Today you will focus on the interior and exterior angles of regular polygons.

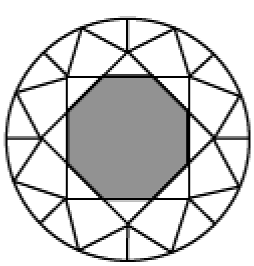
As you work today, keep the following focus questions in mind:

Does it matter if the polygon is regular?

Is there another way to find the answer?

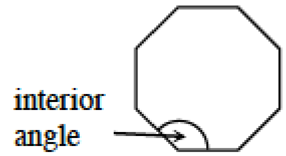
What's the connection?

**8-24.** Diamonds, a very valuable naturally-occurring gem, have been popular for centuries because of their beauty, durability, and ability to reflect a spectrum of light. In 1919, a diamond cutter from Belgium, Marcel Tolkowsky, used his knowledge of geometry to design a new shape for a diamond, called the "round brilliant cut" (top view shown below). He discovered that when diamonds are carefully cut with flat surfaces (called facets or faces) in this design, the angles maximize the brilliance and reflective quality of the gem.



Notice that at the center of this design is a regular octagon with equal sides and equal interior angles. For a diamond cut in this design to achieve its maximum value, the octagon must be cut carefully and accurately. One miscalculation, and the value of the diamond can be cut in half!

a. Determine the measure of each interior angle of a regular octagon. Explain how you found your answer.




b. What about the interior angles of other regular polygons? Find the interior angles of a regular nonagon and a regular 100-gon.

c. Will the process you used for part (a) work for any regular polygon? Write an expression that will calculate the interior angle of an *n*-gon.

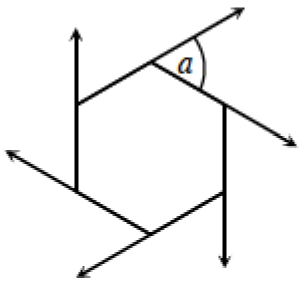
**8-25.** Fern states, "*If a triangle is equilateral, then all angles have equal measure and it must be a regular polygon.*" Does this reasoning work for polygons with more than three sides? Investigate this idea below.

a. If all of the sides of a polygon, such as a quadrilateral, are equal, does that mean that the angles must be equal? If you can, draw a counterexample.

b. What if all of the angles are equal? Does that force a polygon to be equilateral? Explain your thinking. Draw a counterexample on your paper, if possible.

**8-26.** Jeremy asks, "*What about exterior angles? What can we learn about them?"*

a. Examine the regular hexagon shown below. Angle *a* is an example of an **exterior angle** because it is formed on the outside of the hexagon by extending one of its sides. Are all of the exterior angles of a regular polygon equal? Explain how you know.




b. Find *a*. Be prepared to share how you found your answer.

c. This regular hexagon has six exterior angles, as shown in the diagram above. What is the sum of the exterior angles of a regular hexagon?

d. What can you determine about the exterior angles of other regular polygons? Explore this with your team. Have each team member choose a different shape from the list below to analyze. For each shape:

- Find the measure of one exterior angle of that shape, and
- Find the sum of the exterior angles.

| | |
|---|---|
| 1. equilateral triangle | 2. regular octagon |
| 3. regular decagon | 4. regular dodecagon (12-gon) |

e.

Compare your results from part (d). As a team, write a conjecture about the sum of the exterior angles of polygons based on your observations. Be ready to share your conjecture with the rest of the class.

f. Is your conjecture from part (e) true for all polygons or for only regular polygons? Does it matter if the polygon is convex? Explore these questions using Exterior Angles: Triangles; Quadrilaterals; Pentagons; Hexagons (Desmos) geometric eTools or use the Lesson 8.1.3 Resource Page and tracing paper from your teacher. Write a statement explaining your findings.

**8-27.** Use your understanding of polygons to answer the questions below, if possible. If there is no solution, explain why not.

a. Gerardo drew a regular polygon that had exterior angles measuring 40°. How many sides did his polygon have? What is the name for this polygon?

b. A polygon has an interior angle sum of 2520°. How many sides does it have?

c. A quadrilateral has four sides. What is the measure of each of its interior angles?

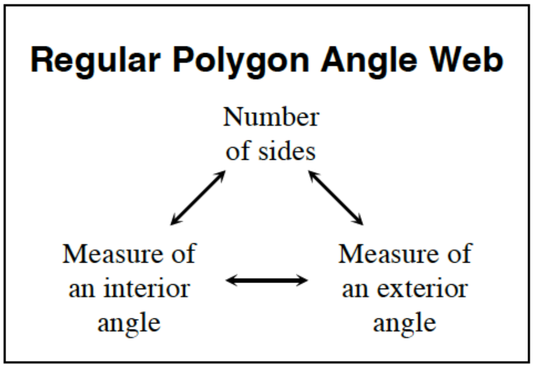
d. What is the measure of an interior angle of a regular 360-gon? Is there more than one way to find this answer?

**8-28.** LEARNING LOG

How can you find the interior angle of a regular polygon? What is the sum of the exterior angles of a polygon? Write a Learning Log entry about what you learned during this lesson. Title this entry “Interior Angles and Sum of Exterior Angles of a Polygon” and include today’s date.

# 8.1.4 Is there another way?

## Regular Polygon Angle Connections




During Lessons 8.1.1 through 8.1.3, you have discovered several ways the number of sides of a regular polygon is related to the measures of the interior and exterior angles of the polygon. These relationships can be represented in the diagram below.

How can these relationships be useful? What is the most efficient way to go from one measurement to another? This lesson will explore these questions so that you will have a complete set of tools to analyze the angles of a regular polygon.

**8-36.** Which connections in the Regular Polygon Angle Web do you already have? Which do you still need? Explore this as you answer the questions below.

a. If you know the number of sides of a regular polygon, how can you find the measure of an interior angle directly? Find the measurements of an interior angle of a 15-gon.

b. If you know the number of sides of a regular polygon, how can you find the measure of an exterior angle directly? Find the measurements of an exterior angle of a 10-gon.

c. What if you know that the measure of an interior angle of a regular polygon is 162°? How many sides must the polygon have? Show all work.

d. If the measure of an exterior angle of a regular polygon is 15°, how many sides does it have? What is the measure of an interior angle? Show how you know.

**8-37.** Suppose a regular polygon has an interior angle measuring 120°. Find the number of sides using *two* different strategies. Show all work. Which strategy was most efficient?

**8-38.** Use your knowledge of polygons to answer the questions below, if possible.

a. How many sides does a polygon have if the sum of the measures of the interior angles is 1980°? 900°?

b. If the exterior angle of a regular polygon is 90°, how many sides does it have? What is another name for this shape?

c. Each interior angle of a regular pentagon has measure $2x + 4°$. What is $x$? Explain how you found your answer.

d. The measures of four of the exterior angles of a pentagon are 57°, 74°, 56°, and 66°. What is the measure of the remaining angle?

e. Find the sum of the interior angles of an 11-gon. Does it matter if it is regular or not?

**8-39.** LEARNING LOG

In a Learning Log entry, copy the Regular Polygon Angle Web that your class created. Explain what it represents and give an example of at least two of the connections. Title this entry “Regular Polygon Angle Web” and include today’s date.

# 8.1.5 What is the area?

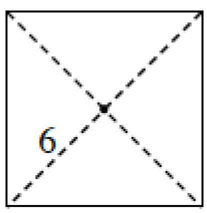
## Finding Areas of Regular Polygons

In Lesson 8.1.4, you developed a method to find the measures of the interior and exterior angles of a regular polygon. How can this be useful? Today you will use what you know about the angles of a regular polygon to explore how to find the area of any regular polygon with *n* sides.

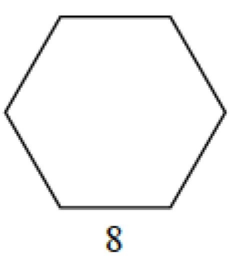
**8-47.** USING MULTIPLE STRATEGIES

With your team, find the area of each shape below *twice*, each time using a distinctly different method or strategy. Make sure that your results from using different strategies are the same. Be sure that each member of your team understands each method.

a. square




b. regular hexagon



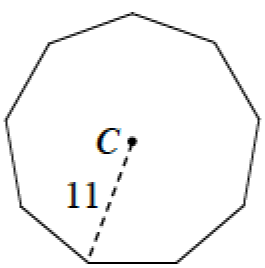
**8-48.** Create a presentation that shows the two different methods that your team used to find the area of the regular hexagon in part (b) of problem 8-47.

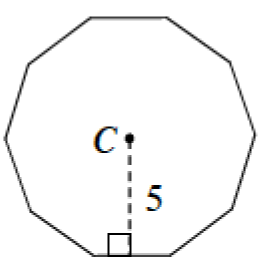
Then, as you listen to other teams present, look for strategies that are different than yours. For each one, consider the questions below.

- *Which geometric tools does this method use?*

- *Would this method help find the area of other regular polygons (like a pentagon or 100-gon)?*

**8-49.** Which method presented by teams in problem 8-48 seemed able to help find the area of other regular polygons? Discuss this with your team. Then find the area of the two regular polygons below. If your method does not work, switch to a different method. Assume *C* is the center of each polygon.

a.


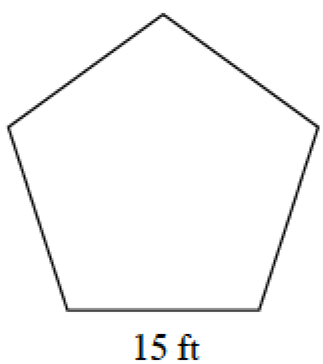

b.


**8-50.** LEARNING LOG

So far, you have found the area of a regular hexagon, nonagon, and decagon. How can you calculate the area of *any* regular polygon? Write a Learning Log entry describing a general process for finding the area of a polygon with *n* sides. Title this entry "Area of a Regular Polygon" and label it with today's date.

**8-51.** Beth needs to fertilize her flowerbed, which is in the shape of a regular pentagon. A bag of fertilizer states that it can fertilize up to 150 square feet, but Beth is not sure how many bags of fertilizer she should buy.

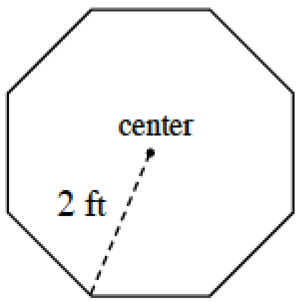
Beth does know that each side of the pentagon is 15 feet long. Copy the diagram of the regular pentagon below onto your paper. Find the area of the flowerbed and tell Beth how many bags of fertilizer to buy. Explain how you found your answer.



**8-52.** GO, ROWDY RODENTS!

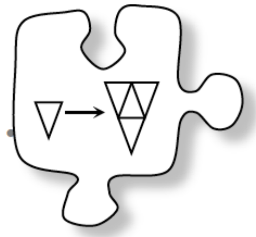
Recently, your school ordered a stained-glass window with the design of the school's mascot, the rodent. Your student body has decided that the shape of the window will be a regular octagon, shown below. To fit in the space, the window must have a radius of 2 feet. The **radius of a regular polygon** is the distance from the center to each vertex.

a. A major part of the cost of the window is the amount of glass used to make it. The more glass used, the more expensive the window. Your principal has turned to your class to determine how much glass the window will need. Copy the diagram onto your paper and find its area. Explain how you found your answer.




b. The edge of the window will have a polished brass trim. Each foot of trim will cost $48.99. How much will the trim cost? Show all work.

# 8.2.1 How does the area change?



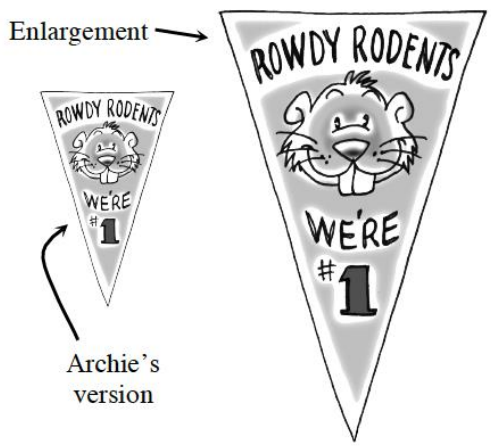
## Area Ratios of Similar Figures

Much of this course has focused on similarity. In Chapter 3, you investigated how to enlarge and reduce a shape to create a similar figure. You also have studied how to use proportional relationships to find the measures of sides of similar figures. Today you will study how the areas of similar figures are related. That is, as a shape is enlarged or reduced in size, how does the area change?

**8-67.** MIGHTY MASCOT

To celebrate the victory of your school's championship girls' ice hockey team, the student body has decided to hang a giant flag with your school's mascot on the gym wall.

To help design the flag, your friend Archie has created a scale version of the flag measuring 1 foot wide and 1.5 feet tall.




a. The student body thinks the final flag should be 3 feet tall. How wide would this enlarged flag be? Justify your solution.

b. If Archie used $2 worth of cloth to create his scale model, then how much will the cloth cost for the full-sized flag? Discuss this with your team. Explain your reasoning.

c. Obtain the Lesson 8.2.1A Resource Page and scissors from your teacher. Carefully cut enough copies of Archie's scale version to fit into the large flag. How many did it take? Does this confirm your answer to part (b)? If not, what will the cloth cost for the flag?

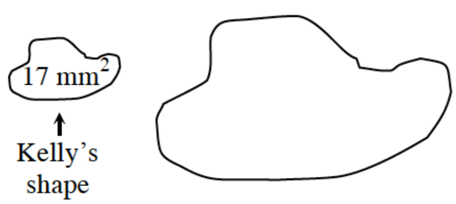
d. The student body is reconsidering the size of the flag. It is now considering enlarging the flag so that it is 3 or 4 times the width of Archie's model. How much would the cloth for a similar flag that is 3 times as wide as Archie's model cost? What if the flag is 4 times as wide?

To answer this question, first *estimate* how many of Archie's drawings would fit into each enlarged flag. Then obtain one copy of the Lesson 8.2.1B Resource Page for your team and confirm each answer by fitting Archie's scale version into the enlarged flags.

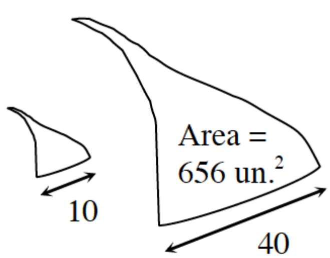
**8-68.** Write down any observations or patterns you found while working on problem 8-67. For example, if the area of one shape is 100 times larger than the area of a similar shape, then what is the ratio of the corresponding sides (also called the **linear scale factor**)? And if the linear scale factor is *r*, then how many times larger is the area of the new shape?

**8-69.** Use your pattern from problem 8-68 to answer the following questions.

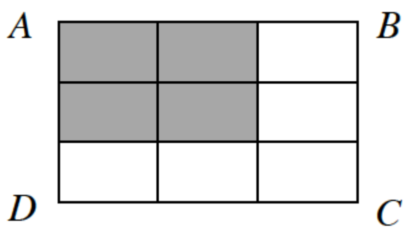
a. Kelly's shape below has an area of 17 mm$^2$. If she enlarges the shape with a linear scale (zoom) factor of 5, what will be the area of the enlargement? Show how you got your answer.




b. Examine the two similar shapes below. What is the linear scale factor? What is the area of the smaller figure?

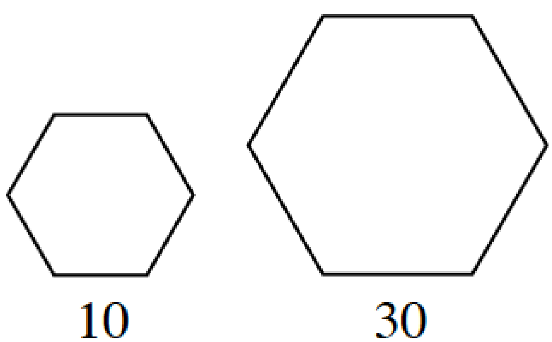


c. Rectangle *ABCD* at right is divided into nine smaller congruent rectangles. Is the shaded rectangle similar to *ABCD*? If so, what is the linear scale factor? And what is the ratio of the areas? If the shaded rectangle is not similar to *ABCD*, explain how you know.




d. While ordering carpet for his rectangular office, Trinh was told by the salesperson that a 16' by 24' piece of carpet costs $800. Trinh then realized that he read his measurements wrong and that his office is actually 8' by 12'. *"Oh, that's no problem,"* said the salesperson. *"That is half the size and will cost $400 instead."* Is that fair? Decide what the price should be.

**8-70.** If the side length of a hexagon triples, how does the area increase? First make a prediction using your pattern from problem 8-68. Then confirm your prediction by calculating and comparing the areas of the two hexagons shown below.



# 8.2.2 How does the area change?

## Ratios of Similarity

Today you will continue investigating the ratios between similar figures. As you solve today's problems, look for connections between the ratios of similar figures and what you already know about area and perimeter.

**8-78.** TEAM PHOTO

Alice has a 4-inch by 5-inch photo of your school's championship girls' ice hockey team. To celebrate their recent victory, your principal wants Alice to enlarge her photo for a display case near the main office.

a. When Alice went to the print shop, she was confronted with many choices of sizes: 7-inch by 9-inch, 8-inch by 10-inch, and 12-inch by 16-inch.

   She's afraid that if she picks the wrong size, part of the photo will be cut off. Which size should Alice pick and why?

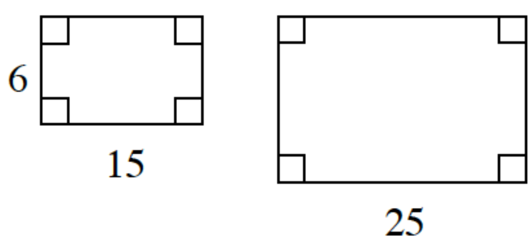
b. The cost of the photo paper to print Alice's 4-inch-by-5-inch picture is \$0.45. Assuming that the cost per square inch of photo paper remains constant, how much should it cost to print the enlarged photo? Explain how you found your answer.

c. Unbeknownst to her, the vice-principal also went out and ordered an enlargement of Alice's photo. However, the photo paper for his enlargement cost \$7.20! What are the dimensions of his photo?

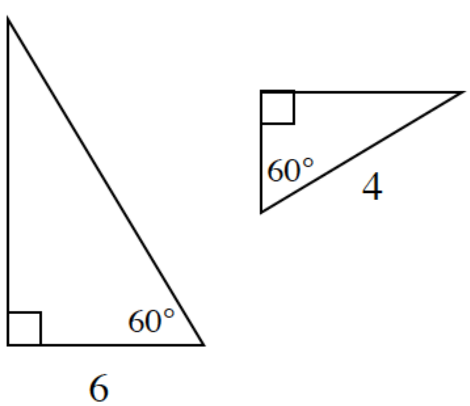
**8-79.** So far, you have discovered and used the relationship between the areas of similar figures. How are the perimeters of similar figures related? Confirm your intuition by analyzing the pairs of similar shapes below. For each pair, calculate the areas and perimeters and complete a table like the one shown below. To help see patterns, reduce fractions to lowest terms or find the corresponding decimal values.

| | Ratio of Sides | Perimeter | Ratio of Perimeters | Area | Ratio of Areas |
|---|---|---|---|---|---|
| small figure | | | | | |
| large figure | | | | | |
| | | | | | |
| | | | | | |

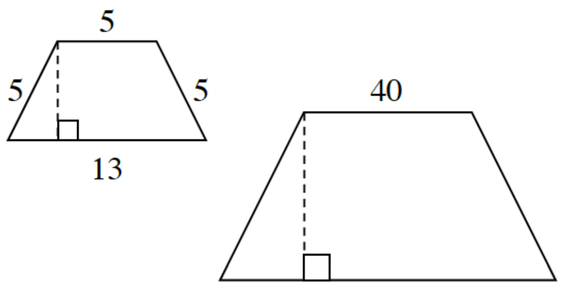
a.



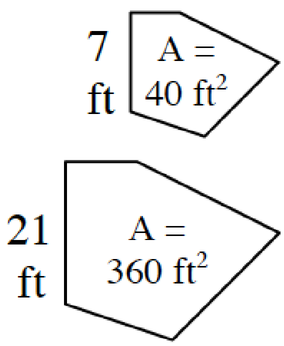

b.



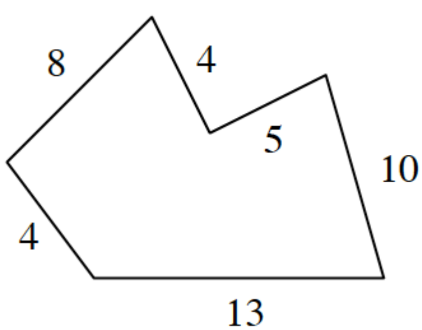
c.




**8-80.** While Jessie examines the two figures below, she wonders if they are similar. Decide with your team if there is enough information to determine if the shapes are similar. Justify your conclusion.




**8-81.** Your teacher enlarged the figure below so that the area of the similar shape is 900 square cm. What is the perimeter of the enlarged figure? Be prepared to explain your method to the class.




**8-82.** LEARNING LOG

Reflect on what you have learned in Lessons 8.2.1 and 8.2.2. Write a Learning Log entry that explains what you know about the areas and perimeters of similar figures. What connections can you make with other geometric concepts? Be sure to include an example. Title this entry "Areas and Perimeters of Similar Figures" and include today's date.

# 8.3.1 What if it has infinitely many sides?

## A Special Ratio

In Section 8.1, you developed a method to find the area and perimeter of a regular polygon with *n* sides. You carefully calculated the area of regular polygons with 5, 6, 8, and even 10 sides. But what if the regular polygon has an infinite number of sides? How can you predict its area?

As you investigate this question today, keep the following focus questions in mind:

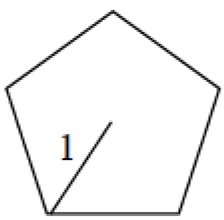
What's the connection?

Do I see any patterns?

How are the shapes related?

**8-90.** POLYGONS WITH INFINITE SIDES

In order to predict the area and perimeter of a polygon with infinitely many sides, your team is going to work with other teams to generate data in order to find a pattern.




Your teacher will assign your team three of the regular polygons below. For each polygon, find the area and perimeter if the radius is 1 (as shown in the diagram of the regular pentagon above). Leave your answer accurate to the nearest 0.01. Place your results into a class chart to help predict the area and perimeter of a polygon with infinitely many sides.

a. equilateral triangle

d. square

g. regular pentagon

j. regular hexagon

b. regular octagon

e. regular nonagon

h. regular decagon

k. regular 15-gon

c. regular 30-gon

f. regular 60-gon

i. regular 90-gon

l. regular 180-gon

**8-91.** ANALYSIS OF DATA

With your team, analyze the chart created by the class.

a. What do you predict the area will be for a regular polygon with infinitely many sides? What do you predict its perimeter will be?

b. What is another name for a regular polygon with infinitely many sides?

c. Does the number 3.14… look familiar? If so, share what you know with your team. Be ready to share your idea with the class.

**8-92.** LEARNING LOG

Record the area and circumference of a circle with radius 1 in your Learning Log. Then, include a brief description of how you “discovered” $\pi$. Title this entry “Pi” and include today’s date.

# 8.3.2 What is the relationship?

## Area and Circumference of a Circle

In Lesson 8.3.1, your class discovered that the area of a circle with radius 1 unit is $\pi$ units$^2$ and that the circumference is $2\pi$ units. But what if the radius of the circle is 5 units or 13.6 units? Today you will develop a method to find the area and circumference of circles when the radius is not 1. You will also explore parts of circles (called sectors and arcs) and learn about their measurements.

As you and your team work together, remember to ask each other questions such as:

Is there another way to solve it?

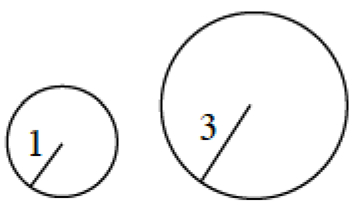
What's the relationship?

What is area? What is circumference?

**8-100.** AREA AND CIRCUMFERENCE OF A CIRCLE

Now that you know the area and circumference (perimeter) of a circle with radius 1, how can you find the area and circumference of a circle with any radius? Explore with the [8-100 Student eTool] (Desmos).

a. First examine how the circles below are related. Since circles always have the same shape, what is the relationship between any two circles?




b. What is the ratio of the circumferences (perimeters)? What is the ratio of the areas? Explain.

c. If the area of a circle with radius of 1 is $\pi$ square units, what is the area of a circle with radius 3 units? With radius 10 units? With radius *r* units?

d. Likewise, if the circumference (perimeter) of a circle is $2\pi$ units, what is the circumference of a circle with radius 3? With radius 7? With radius $r$?

**8-101.** Read the definitions of radius and diameter in the Math Notes box for this lesson. Then answer the questions below.

a. Find the area of a circle with radius 10 units.

b. Find the circumference of a circle with diameter 7 units.

c. If the area of a circle is $121\pi$ square units, what is its diameter?

d. If the circumference of a circle is $20\pi$ units, what is its area?

**8-102.** The giant sequoia trees in California are famous for their immense size and old age. Some of the trees are more than 2500 years old and tourists and naturalists often visit to admire their size and beauty. In some cases, you can even drive a car through the base of a tree!

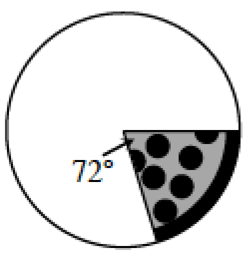
One of these trees, the General Sherman tree in Sequoia National Park, is the largest living thing on the earth. The tree is so gigantic, in fact, that the base has a circumference of 102.6 feet! Assuming that the base of the tree is circular, how wide is the base of the tree? That is, what is its diameter? How does that diameter compare with the length and width of your classroom?

**8-103.**To celebrate their victory, the girls' ice-hockey team went out for pizza.

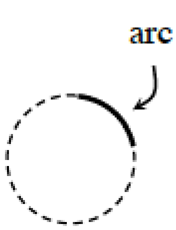
a. The goalie ate half of a pizza that had a diameter of 20 inches! What was the area of pizza that she ate? What was the length of crust that she ate? Leave your answers in exact form. That is, do not convert your answer to decimal form.

b. Sonya chose a slice from another pizza that had a diameter of 16 inches.
If her slice had a central angle of 45°, what is the area of this slice? What is the length of its crust? Show how you got your answers.

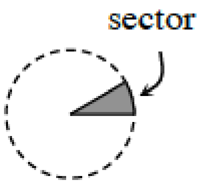
c. As the evening drew to a close, Sonya noticed that there was only one slice of the goalie's pizza remaining. She measured the central angle and found out that it was 72°. What is the area of the remaining slice? What is the length of its crust? Show how you got your answer.




d. A portion of a circle (like the crust of a slice of pizza) is called an **arc**. This is a set of connected points a fixed distance from a central point. The length of an arc is a part of the circle's circumference. If a circle has a radius of 6 cm, find the length of an arc with a central angle of 30°.



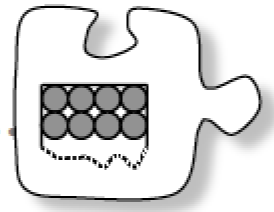
e. A region that resembles a slice of pizza is called a **sector**. It is formed by two radii of a central angle and the arc between their endpoints on the circle. If a circle has radius 10 feet, find the area of a sector with a central angle of 20°.




**8-104.** LEARNING LOG

Reflect on what you have learned today. How did you use similarity to find the areas and circumferences of circles? How are the radius and diameter of a circle related? Write a Learning Log entry about what you learned today. Title this entry "Area and Circumference of a Circle" and include today's date.

# 8.3.3 How can I use it?



## Circles in Context

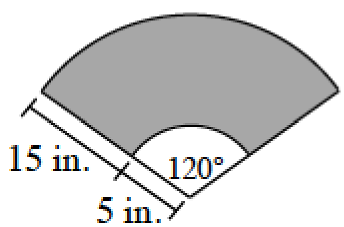
In Lesson 8.3.2, you developed methods to find the area and circumference of a circle with radius *r*. During this lesson, you will work with your team to solve problems from different contexts involving circles and polygons.

**8-112.** While the earth's orbit (path) about the sun is slightly elliptical, it can be approximated by a circle with a radius of 93,000,000 miles.

a. How far does the earth travel in one orbit about the sun? That is, what is the approximate circumference of the earth's path?

b. Approximately how fast is the earth traveling in its orbit in space? Calculate your answer in miles per hour.

**8-113.** A certain car's windshield wiper clears a portion of a sector as shown shaded below. If the angle the wiper pivots during each swing is 120°, find the area of the windshield that is wiped during each swing.



**8-114.** THE GRAZING GOAT

Zoe the goat is tied by a rope to one corner of a 15 meter-by-25 meter rectangular barn in the middle of a large, grassy field. Over what area of the field can Zoe graze if the rope is:

a. 10 meters long?

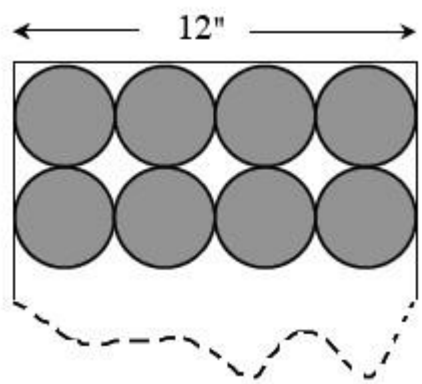
b. 20 meters long?

c. 30 meters long?

d. Zoe is happiest when she has at least 400 m$^2$ to graze. What possible lengths of rope could be used?

**8-115.** THE COOKIE CUTTER

A cookie baker has an automatic mixer that turns out a sheet of dough in the shape of a square 12 inches wide. His cookie cutter cuts 3-inch diameter circular cookies as shown at right. The supervisor complained that too much dough was being wasted and ordered the baker to find out what size cookie would have the least amount of waste.



**Your Task:**

Analyze this situation and determine how much cookie dough is "wasted" when 3-inch cookies are cut. Then have each team member find the amount of dough wasted when a cookie of a different diameter is used. Compare your results.

Write a note to the supervisor explaining your results. Justify your conclusion.